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# Law of proliferation of periodic orbits in pseudointegrable billiards

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We prove that the periodic-orbit counting function, a measure of the rate of proliferation of periodic orbits, for a barrier billiard and the $\pi/3$-rhombus billiard is of the form $ax^2+bx+c$, where $x$ is the length (equivalently, period) up to which periodic orbits are counted and $a,b,c$ are system-specific constants. The generality of our arguments strongly suggests that the law of proliferation given here is a representation of general truth about two-dimensional plane-polygonal billiards.

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The dynamics of nonintegrable systems is one of the central areas of research in recent times [1]. An immense amount of work has been done on classically chaotic systems and their quantal counterparts. Billiards have served as model examples in our understanding of various shades of behaviors between complete regularity and chaos. Although a reasonably good understanding has been achieved for completely chaotic and integrable systems, the systems exhibiting intermediate behaviors have only recently been studied in detail [2,3]. To study the behavior of nonintegrable systems, it is natural to examine the systems closest to the integrable systems. Particles enclosed in polygonal boxes (polygonal billiards) constitute examples of this kind. The well-known integrable system in this class of systems is a particle in a square box. The invariant integral surface in the phase space is topologically equivalent to a torus (sphere with one handle; genus, $g=1$). A simplest deformation of this square billiard is a rhombus billiard. The most well studied rhombus billiard is the $\pi/3$-rhombus billiard [2,3]. Yet another simple example of the nonintegrable billiard is a barrier billiard wherein there is a linear barrier of length $L/2$ placed parallel to one of the sides in the center of a square box (side length, $L$). In both these systems, any typical trajectory lies on a surface topologically equivalent to a double torus ($g=2$). This implies that the billiards under present discussion are nonintegrable; however, the deviation from integrability is only marginal. Due to this fact, these systems are termed almost integrable [4] ($A$-integrable) or pseudointegrable [5].

An extensive study of the solutions of the $\pi/3$-rhombus billiard reveals that the pure rhombus modes are quite irregular [2] and the spectral statistics deviates from Poisson-like to Wigner-like. The semiclassical analysis of the spectral statistics rests on the principle of uniformity [6,7]. For a good understanding of the principle of uniformity, one needs to establish the law determining the rate of proliferation of the periodic orbits for this class of systems.

For chaotic dynamical systems, it was proved that the periodic orbits proliferate exponentially with the period of the orbit [8,9]. For the $A$-integrable system, it was conjectured by Katok [10], and independently shown by Gutkin [11], that the asymptotic law would be exactly quadratic. This result was recently sharpened by an exact analysis on the $\pi/3$-rhombus billiard [3]. The aim of this Rapid Communication is to extend these results to the barrier billiard considered by Hannay and McCraw [12]. On the basis of our general arguments, we believe that the results obtained for these systems represent a general truth about polygonal billiards. Of course, the exact law of proliferation for a system exhibiting a general intermediate behavior stands as an open problem (e.g., as a function of the genus of the invariant surface).

By a family of periodic orbit, we mean an isolated trajectory closing after an odd number of reflections, or a band of trajectories closing after an even number of reflections. The number of families of periodic orbits of length less than or equal to $x$, $F(x)$, is finite for any $x$ [11]. For a polygon, it was conjectured [10] that

$$F(x)=cx^n+O(x^{n-1}) . \tag{1}$$

For $A$-integrable polygons, $n=2$. If we consider an integrable system, $\Delta$ corresponding to an $A$-integrable system, $P$, and let $g$ be the genus of the surface $R$ corresponding to $P$, denoting by $|\Delta|$ and $|P|$ the respective areas of $\Delta$ and $P$, Gutkin [11] proved that there exists a constant $c_1$ such that

$$F(x)=c_1(\pi g x^2/|P|)+O(x) . \tag{2}$$

The constant $c_1\in[1,|P|/|\Delta|]$. It was shown later [3] that the bounds on $c_1$ must go down by two orders of magnitude; in fact, $c_1\sim 10^{-2}$. For the $\pi/3$-rhombus billiard, we have shown that $c_1=53/108\pi^2$. We come to an explanation of the discrepancy in the quadratic coefficient later. For the $\pi/3$-rhombus billiard, an exact number-theoretic analysis was carried out and we obtained [3]

$$F(x)=(53\sqrt{3}/81\pi)x^2/L^2+[26(4\sqrt{3}-3)/81\pi]x/L + 12(2\sqrt{3}-3)/27\pi . \tag{3}$$

This analytic result is asymptotically exact when compared with the numerically enumerated orbits (Fig. 1). Also, it is worth mentioning that the results match, even for $x\sim 10^1$.

We now present our calculation for the Hannay-McCraw (HM) billiard [12]. By stacking the domains of the billiard side by side in both the orthogonal directions, one obtains an infinite lattice of barriers and gaps, with barrier to gap to ratio unity. One can label the end points of barriers by integer pairs that form lattice points. It can be easily seen that the straightened version of a rational gradient ($=|p/q|$) trajectory will initially meet lattice point $(q,p)$ and then repeat itself by meeting lattice points $(mq,mp)$, where $m\in\mathbb{Z}$. On the other hand, the irrational gradient trajectory will never visit any lattice point, though it will come arbitrarily close to many lattice points, hence will never be periodic. Thus the periodic orbits in the system are the ones that hit any lattice point $(q,p)$ in this array of barriers; the gradient of such trajectories will be given by $|p/q|$. By the above arguments, we need to consider only the pairs $(q,p)$ such that $q$ and $p$ are coprime, since they only give a primitive periodic orbit, and points $(mq,mp)$ where $m\in\mathbb{Z}$, give $m$ repetitions of a primitive periodic orbit corresponding to $(q,p)$. Each such $(q,p)$ gives different numbers of bands or families of periodic orbits, depending on whether the pair is odd-odd $(o,o)$, even-odd $(e,o)$, or odd-even $(o,e)$. The length of periodic orbit in a given family corresponding to a lattice point $(q,p)$ is given by $l=cL(q^2+p^2)^{1/2}$, where $c$ depends on the number of families or periodic orbits. It can be seen that [8], for $(q,p)$,

$$c=\begin{cases}4 \ \ [\text{closing at } (4q,4p), \text{one band}] \ \text{ for } (o,o)\\ 2,2 \ \ [\text{two bands, each closing at } (2q,2p)] \ \text{ for } (o,e)\\ 1,1,2 \ \ [\text{two bands closing at } (q,p) \text{ and one at } (2q,2p)] \ \text{ for } (e,o)\ .\end{cases} \tag{4}$$

If $1\le x$ for a given family of points $(q,p)$, the contribution from this family of $(q,p)$ should be counted in $F(x)$. Drawing a quarter circle of radius $l_1$ in the quadrant under consideration, all points $(q,p)$ having a family of periodic orbits with length $cl_1\le x$ (or $l_1\le x/c$) must be considered for the calculation of $F(x)$. The quarter circle is inscribed in a square $OABC$ with side length $l_1$ (Fig. 2). Thus the area of this square is $l_1^2$ and the area of the quarter circle $OAC$ is $\pi l_1^2/4$. We shall denote the integer (fractional) part of a number by [ ] ({ }). The number of lattice points $N$ in the square $OABC$ is given by $([l_1/L]+1)^2$ $((l_1/L-\{l_1/L\}+1)^2)$. Taking the fractional part of $l_1/L$, on an average as $\frac{1}{2}$, we can write

$$N=(l_1/L)^2+(l_1/L)-1/4\ . \tag{5}$$

Then the number of lattice points in a quarter circle is just

$$N_q=N(\pi l_1^2/4)/l_1^2=\pi N/4\ . \tag{6}$$

Since the probability that two randomly chosen numbers are coprime is $(6/\pi^2)$ [13], the number of coprime lattice points in a quarter circle is

$$N_C=(6/\pi^2)(\pi N/4)$$
$$=(3/2\pi)l_1^2/L^2+(3/2\pi)l_1/L+3/8\pi\ . \tag{7}$$

The counting function can now be written explicitly as [using Eq. (4)]

$$F(x)=P_{oo}N_{oo}(x/4)+2P_{oe}N_{oe}(x/2)$$
$$+2P_{eo}N_{eo}(x)+P_{eo}N_{eo}(x/2)\ , \tag{8}$$

where, e.g., $P_{oo}$ is the probability that a given coprime lattice point is of (odd,odd) type and $N_{oo}(x)$ is the total number of odd-odd coprime lattice points contained in the quarter circle of radius $x$ ($=N_C$). Trivially,

$$P_{oo}=P_{oe}=P_{eo}=\tfrac{1}{3}\ , \tag{9}$$

thus

$$F(x)=\tfrac{1}{3}[N_C(x/4)+3N_C(x/2)+2N_C(x)]$$
$$=(45/32\pi)(x/L)^2+(15/8\pi)(x/L)+3/4\pi\ . \tag{10}$$

This is the asymptotic law of proliferation of periodic orbits for the system under consideration. How fast the actual $F(x)$ converges to Eq. (10) depends on the rate of convergence of $P_{oo}$, $P_{oe}$, $P_{eo}$, and $N_C$ in accordance with Eqs. (9) and (7), respectively. It can be easily seen that $P_{oo}, P_{oe}, P_{eo}$ converges rapidly to $\frac{1}{3}$. Our numerical results show that the percentage difference between the actual

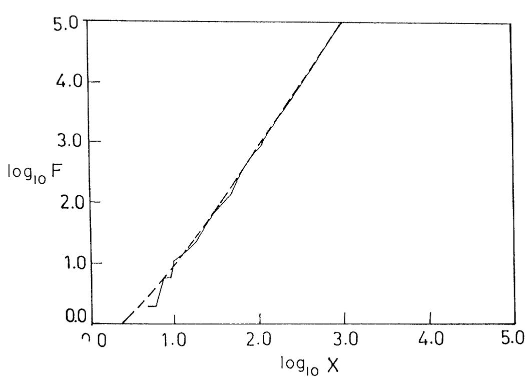


FIG. 1. A log-log plot of the counting functions, $F(x)$ vs the length $(x)$ of primitive periodic orbit. The dashed curve represents Eq. (3) and the solid curve represents $F(x)$ obtained numerically.

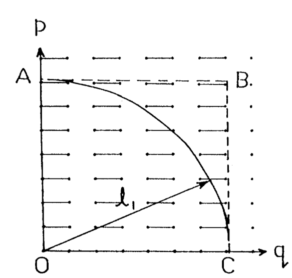


FIG. 2. By taking the unit cell we obtain a periodic array of barriers. The end points of the barriers can be assigned integer labeling.

number of coprime pairs and the results obtained by Eq. (7) decreases very fast as the $x$ increases (even at $x=50$ and 500 the percentage difference is only 2% and 0.19%, respectively). For similar reasons we get an equally remarkable agreement in the case of the $\pi/3$-rhombus billiard, as seen in Fig. 1.

Let us look at the reason underlying the difference between the quadratic coefficient obtained by us and the one by Gutkin. As is clear from Eqs. (3) and (10), the quadratic coefficient is one order of magnitude less than the Gutkin estimate in the case of the HM barrier billiard. In considering the number of lattice points formed by stacking the fundamental region of the corresponding integrable system, the condition of coprimality was not taken into account by Gutkin. As explained earlier, an orbit labeled by a pair $(mq,mp)$, where $m\in\mathbb{Z}$, gives $m$ repetitions of a primitive periodic orbit corresponding to point $(q,p)$, where $q$ and $p$ are coprime. Hence ignoring the coprimality condition leads to an over counting of the periodic orbits. Further, due to symmetry in the tesselated two-dimensional plane, calculations need to be performed for the $\pi/4$ sector in the HM billiard and for the $\pi/6$ sector in the $\pi/3$-rhombus billiard. In general, of course, for a domain with a discrete symmetry of order $N$, only a $\pi/N$ sector needs consideration. Finally, one must note a basic difference between the lattice generated by the fundamental polygonal billiard and the corresponding integrable system, which lies in the incomplete tesselation of the plane by the nonintegrable billiards. For instance, the barriers are of zero width and finite length in the two examples considered in this paper. It is this structure that enables us to completely classify the orbits via integer labeling. The relative weights [$P_{oo},P_{oe},P_{eo}$ in the HM billiard, and $P_{oocc},P_{oocc}$, etc. (see Ref. [3]) in the $\pi/3$-rhombus billiard] for different types of coprime lattice points differ in different systems and lead to a different quadratic coefficient. Hence, to give a general formula for the law of proliferation of periodic orbits exactly demands a complete enumeration and classification of periodic orbits. Although this important question cannot be answered today, we do present a general recipe in the following that comes very close to an exact formula for the quadratic coefficient (see Fig. 3).

Presently, however, we discuss the nature of this law when repetitions are counted. Recently, it was conjectured [14] that

$$F(x)\sim x^{2+\delta},\quad \delta>0\ . \tag{11}$$

This conjecture was tested using numerical calculations [14]. However, it ensues from our arguments in this paper and Ref. [3] that the law is exactly quadratic (asymptotically) and that the asymptotic convergence is very fast ($x\sim 10^1$). The authors of Ref. [14] have not distinguished the primitive periodic orbits from their repetitions. If we follow along lines similar to the arguments given above by considering the repetitions of primitive periodic orbits, we obtain an asymptotic behavior of the counting function, which we will now discuss. It may be remarked that the usefulness of the following calculations emerges from the fact that one requires the primitive periodic orbits and their repetitions for the semiclassical calculations of various measures of the spectral statistics (e.g., $\Delta_3$ statistic, number variance, etc.).

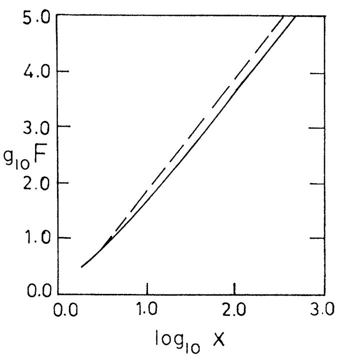


FIG. 3. A log-log plot of (i) the modified counting function, $F'(x)$, taking account of primitive orbits, and its repetitions [see Eq. (15)] represented by the dashed curve; and (ii) the counting function $F(x)$, taking account of only primitive orbits for the Hannay-McCraw billiard [see Eq. (10)], represented by a solid curve, vs the length $(x)$ of the primitive periodic orbit.

If we are counting repetitions of the primitive orbits of length $\le x$, then the primitive orbits of the length between $x$ and $x/2$ will not be repeated; the primitive orbits with length $\ge x/2$ but $<x/3$ will be repeated once; the primitive orbits with length $\ge x/3$ but $<x/4$ will be repeated twice, and so on. Thus taking account of these repetitions, one can write an expression for the number of "effective" coprime lattice points within a quarter circle of radius $x, N_r(x)$ as

$$N_r(x)=N_C(x)-N_C(x/2)+2[N_C(x/2)-N_C(x/3)] + \cdots +n[N_C(x/n)-N_C(1)]\ . \tag{12}$$

Here $n$ is the largest integer less than $x$; we have neglected $N(l)$ $(l<1)$ since there are no periodic orbits of length less than 1 in the system we have considered. Equation (12) can be rewritten as

$$N_r(x)=N_C(x)+N_C(x/2)+N_C(x/3)+\cdots+N_C(x/n) -nN_C(1)\ . \tag{13}$$

Then, the modified counting function $F'(x)$ becomes

$$F'(x)=\tfrac{1}{3}[N_r(x/4)+3N_r(x/2)+2N_r(x)]\ . \tag{14}$$

Substituting Eq. (13) in Eq. (14), we get

$$F'(x)=\tfrac{1}{3}[(3l^2/2\pi)A(n_4)/16+3A(n_2)/4+2A(n)] +(3l/2\pi)[B(n_4)/4+3B(n_2)/2+2B(n)] -(3/\pi)(n_4+n_2+n)\ , \tag{15}$$

where $n_4$ and $n_2$ are largest integers less than $x/4$ and $x/2$, respectively. $A(n)$ and $B(n)$ are given by $\sum_{i=1}^{n}(1/i^2)$ and $\sum_{i=1}^{n}(1/i)$, respectively. Asymptotically $(n\to\infty)$, $A(n)=\pi^2/6$, and $B(n)=\ln(n)+\gamma$, where $\gamma$ is the Euler-Mascheroni constant, equal to $0.577\,215\,7\ldots$ .

The rate of convergence of the actual $F'(x)$ to Eq. (15) depends upon the rate of convergence of the actual $N_r(x)$ to Eq. (3). Our numerical calculations show that the percentage difference between the actual "effective" coprime numbers and those obtained from Eq. (13) at $x=50$ is 5%, which is almost 2.5 times the one observed for $N_C$. It is for this reason that the convergence to the quadratic law [Eq. (15)] is much slower [14] if one considers repetitions.

For lower values of $x$, it was recently conjectured [14] that the counting function goes as $x^{2+\delta}$ for pseudointegrable billiards. Obviously, it does not agree with our findings for the reasons discussed in the above paragraph. It was further believed [14], due to the law being $x^{2+\delta}$ that the spectral rigidity would vary as $L^{1-\delta}$ ($\delta>0$), where $L$ is the averaging length of an interval over the energy spectrum. It is well known that the spectral rigidity for pseudointegrable systems is intermediate between the Poisson and the Gaussian Orthogonal Ensemble [2] (weaker than a linear dependence [15]). Since we have shown for the $\pi/3$-rhombus billiard and the Hannay-McCraw billiard that the counting function is exactly a quadratic law for all practical purposes (meaning thereby that it is the asymptotic behavior of the proliferation law which is of any significance in the semiclassical derivation of spectral statistics [7]), it follows that the conjectured appearance of $\delta$ in the spectral rigidity having its roots in the counting function is unfounded. Moreover, one can easily see that the rate of proliferation for the square billiard (integrable system) is also asymptotically exactly quadratic. For this system, the $\Delta_3(L)=L/15$ [7]. Now, for the same reasons as exist for the two examples discussed in this paper, there is indeed slow convergence of the counting function (with repetitions) to the exact quadratic law. It clearly brings out the fact that the behavior of the counting function at low values of $x$ is of negligible relevance in the semiclassical analysis of the $\Delta_3$ statistic. It needs to be emphasized that even with repetitions, the counting function goes asymptotically as $x^2$ (as against $x^{2+\delta}$), although at rather large values of $x$ ($\sim 500$). Moreover, one can see that $\delta$ is not even a constant number from the calculations in Ref. [14], a fact that invalidates the conjecture.

To conclude, we have shown analytically and with sufficient numerical support that the rate of proliferation of the periodic orbits is (asymptotically) exactly quadratic. The reason underlying the asymptote, $ax^2+bx+c$, to the counting function is clearly related to the tessellation of the two-dimensional plane by the fundamental region of the billiards. It is well known that a rational polygon can periodically tile a surface that is everywhere flat, in the sense of null Gaussian curvature, except at isolated vertex points of singular negative curvature. A periodic structure that tiles the almost everywhere flat surface may consist of several polygons and hence the space can be assigned distinct labels (albeit complicated), taking account of different periodicities, in a spirit similar to the one presented above. Let us consider an irrational billiard with an internal angle $\beta\pi$, where $\beta$ is an irrational number. By the continued fraction expansion of $\beta$, for the $j$th rational convergent of $\beta$, $\epsilon=|\beta-p_j/q_j|$, can be made as small as desired. Hence there exists an "$\epsilon$-close" rational billiard for any irrational billiard. Thus the arguments leading to the distinct labeling of an almost flat surface are applicable to these billiards.

To enumerate distinct primitive periodic orbits, one needs a condition analogous to the coprimality condition required by the two systems discussed above, since out of all lattice points lying on the same line of a given slope only one will give a primitive periodic orbit. Let us denote the probability of the "coprimality condition" to be satisfied by distinct labels by $P_c$. Furthermore, the classification entailing each distinct label will give rise to relative weights in which the orbits will be distributed; let us denote it by $P_j$ [$j$ denotes classes, e.g. (odd,odd), (even,odd), and (odd,even) in the HM billiard]. For a polygon with symmetry group of order $N$, the points to be considered will be restricted to a $\pi/N$ sector. This number, $N_L$, can be written as $\sum_i(\alpha_i x^2+\beta_i x+\gamma_i)/N$, where the summation is over all periodicities and $x$ has usual meaning. Correspondingly, the number of "coprime" points, $N_{LC}(x)$, are $P_c N_L(x)$. For each class of periodic orbit for which the weight is $P_j$ there may be $k$ types of periodic orbits closing at length $\zeta_{kj}x$. With this the counting function can be written as

$$F(x)=\sum_j P_j \sum_k N_{LC}(x/\zeta_{kj}) , \tag{16}$$

and hence the coefficients of $F(x)=ax^2+bx+c$ are

$$a=(P_c/N)\sum_i \alpha_i \sum_j P_j \sum_k \zeta_{jk}^{-2} , \tag{17a}$$

$$b=(P_c/N)\sum_i \beta_i \sum_j P_j \textstyle\sum_k \zeta_{jk}^{-1} , \tag{17b}$$

$$c=(P_c/N)\sum_i \gamma_i \sum_j P_j . \tag{17c}$$

Hence, our analysis strongly suggests that the asymptote to the counting function will always be of the form $ax^2+bx+c$, and convergence to this asymptote will depend on convergence of both $P_c$ and $P_L$. However, the asymptotic law of proliferation of the periodic orbits will be exactly quadratic ($\sim x^2$). The aim of our discussion on the spectral rigidity is to point out that the basic reason underlying its intermediate behavior (between Poisson and GOE) remains unknown. To this effect, work is currently in progress.

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